Lesson 1: Multiplying and Factoring Polynomial Expressions

Student Outcomes
Students use the distributive property to multiply a monomial by a polynomial and understand that factoring reverses the multiplication process.
Students use polynomial expressions as side lengths of polygons and find area by multiplying.
Students recognize patterns and formulate shortcuts for writing the expanded form of binomials whose expanded form is a perfect square or the difference of perfect squares.

Lesson Notes
Central to the concepts in this lesson is A-APR.1, and understanding the system and operations of polynomial expressions, specifically multiplication and factoring of polynomials. Lengths of time suggested for the examples and exercises in this lesson assume that students remember some of what is presented in the examples from work in earlier modules and earlier grades. Students may need more or less time with this lesson than is suggested. The teacher should make decisions about how much time is needed for these concepts based on the needs of their students.

Standards for Mathematical Practice: This lesson asks students to use geometric models to demonstrate their understanding of multiplication of polynomials.

Classwork
Opening Exercise (4 minutes)

Opening Exercise

Write expressions for the areas of the two rectangles in the figures given below.

Blue: ; Orange:

Now write an expression for the area of this rectangle:
How did you find your answer for the second rectangle?

1. Add the two areas in part (a) or, multiply the length of the rectangle by the width. Both give the same final result.

If you find the area by multiplying the total length times total width what property of operations are you using?

2. The Distributive Property

What would be another way to find the total area?

3. Finding the area of the two separate rectangles and adding their area:

Example 1 (3 minutes)

Have students work on this example with a partner or in small groups. Have the groups share their process and their findings and discuss the differences in processes used (if there are any).

Example 1

The total area of this rectangle is represented by . Find expressions for the dimensions of the total rectangle.

[Students may opt to factor only or: or]

In the example above, ______ and ______ are factors of and could represent the dimensions of the rectangle.

and, or and, or and

Note: If students try to use as the common factor for two or more numbers, point out that, while is indeed a factor, factoring out a does not help in finding the factors of an expression. If this issue arises, it may be necessary to discuss the results when factoring out .

Factoring out the Greatest Common Factor (GCF)

Students now revisit factoring out the greatest common factor as was introduced in Grade 6 Module 2.

When factoring a polynomial, we first look for a monomial that is the greatest common factor (GCF) of all the terms of the polynomial. Then we reverse the distribution process by factoring the GCF out of each term and writing it on the outside of the parentheses.

In the example above we factored out the GCF: .

Exercises 1–3 (3 minutes)

For the exercises below have students work with a partner or in small groups to factor out the GCF for each expression.

Exercises 1–3

Factor each by factoring out the Greatest Common Factor:
Discussion—The Language of Polynomials (4 minutes)

Make sure the students have a clear understanding of the following terms and use them appropriately during instruction. The scaffolding suggestion below may be used to help students understand the process of factoring out the GCF. Begin the discussion by reviewing the definition of prime and composite numbers given in the student materials.

Discussion—The Language of Polynomials

A prime number is a positive integer greater than whose only positive integer factors are and itself.

A composite number is a positive integer greater than that is not a prime number.

A composite number can be written as the product of positive integers with at least one factor that is not or itself.

For example, the prime number has only and as its factors. The composite number has factors of , , and ; it could be written as the product .

Factoring is the reverse process of multiplication (through multiple use of the Distributive Property). We factor a polynomial by reversing the distribution process—factoring the GCF out of each term and writing it on the outside of the parentheses. To check whether the polynomial's factored form is equivalent to its expanded form, you can multiply the factors to see if the product yields the original polynomial.

is called a factored form of .

A nonzero polynomial expression with integer coefficients is said to be prime or irreducible over the integers if it satisfies two conditions:

1) it is not equivalent to or , and

2) if the polynomial is written as a product of two polynomial factors, each with integer coefficients, then one of the two factors must be or .

Note: this definition actually specifies prime numbers and their negatives as well (the case when the polynomial has degree ).

For example: is irreducible over the integers.
Given a polynomial in standard form with integer coefficients, let $g$ be the greatest common factor of all of the coefficients. The polynomial is factored completely over the integers when it is written as a product of $g$ and one or more prime polynomial factors, each with integer coefficients.

In the future, we will learn to factor over the rationals and reals.

Example 2 (4 minutes): Multiply Two Binomials

Demonstrate that the product can be found by applying the distributive property (twice) where the first binomial distributes over each of the second binomial’s terms, and relate the result to the area model as was used in Module 1 and as shown below. Note that while the order of the partial products shown corresponds with the well-known “FOIL” method (Firsts, Outers, Inners, Lasts), teachers are discouraged from teaching polynomial multiplication as a procedure or with mnemonic devices such as “FOIL”. Instead, foster understanding by relating the process to the distributive property and the area model.

Since side lengths of rectangles cannot be negative, it is not directly applicable to use the area model for multiplying general polynomials. (We cannot be certain that each term of the polynomial represents a positive quantity.) However, we can use a tabular method that resembles the area model to track each partial product as we use the distributive property to multiply the polynomials.

Example 2: Multiply Two Binomials

Multiply Two Binomials—Using a Table as an Aid

You have seen the geometric area model used in previous examples to demonstrate the multiplication of polynomial expressions for which each expression was known to represent a measurement of length.

Without a context such as length, we cannot be certain that a polynomial expression represents a positive quantity. Therefore, an area model is not directly applicable to all polynomial multiplication problems. However, a table can be used in a similar fashion to identify each partial product as we multiply polynomial expressions. The table serves to remind us of the area model even though it does not represent area.

For example, fill in the table to identify the partial products of $x^2 - 9$. Then write the product of $x^2 - 9$ in standard form.
Multiply Two Binomials—Without the Aid of a Table

Regardless of whether or not we make use of a table as an aid, the multiplying of two binomials is an application of the Distributive Property. Both terms of the first binomial distribute over the second binomial. Try it with \( \text{.} \) In the example below the colored arrows match each step of the distribution with the resulting partial product:

Example 3 (4 minutes): The Difference of Squares

Example 3: The Difference of Squares

Find the product of \( \text{.} \) Use the distributive property to distribute the first binomial over the second.

With the use of a table:

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</tbody>
</table>

Without the use of a table:

Do you think the linear terms will always be opposites when we multiply the sum and difference of the same two terms? Why?

4. Yes, because when we multiply the first term of the first binomial by the last term of the second we get the opposite of what we get when we multiply the second term of the first binomial by the first term of the second.
So is the difference of two perfect squares. Factoring the difference of two perfect squares reverses the process of finding the product of the sum and difference of two terms.

**Exercise 4 (6 minutes)**
The following can be used as a guided practice or as independent practice.

**Exercise 4**
Factor the following examples of the difference of perfect squares.

a. 

b. 

c. 

d. 

e. 

f. 

Point out that any even power can be a perfect square and that is always a square.

**Write a General Rule for Finding the Difference of Squares**
Write in factored form.

**Exercises 5–7 (4 minutes)**
The following exercises may be guided or modeled, depending on how well students did on the previous example.

**Exercises 5–7**
For each of the following differences of squares, factor completely over the integers.

1. 

(Hint: This one will factor twice.)

**Example 4 (4 minutes): The Square of a Binomial**
It may be worthwhile to let students try their hands at finding the product before you show them how. If students struggle to include every step in the process, pause at each step so that they have time to absorb the operations that took place.

Example 4: The Square of a Binomial

To square a binomial, like \((a + b)^2\), multiply the binomial by itself.

Square the following general examples to determine the general rule for squaring a binomial:

1.

Point out that the process used in squaring the binomial is called expanding; in general, expanding means rewriting a product of sums as a sum of products through use of the Distributive Property.

How are the answers to the two general examples similar? How are they different? What is the cause of the difference between the two?

5. Both results are quadratic expressions with three terms. The first and second examples both have an \(a\), \(b\), and \(c\)-term. In Example 2, the \(c\)-term is negative, while it is positive in Example 1. The negative (subtraction) in the binomial’s constant term in Example 2 causes the middle term to be negative.

Exercises 8–9 (3 minutes)

Exercises 8–9

Square the binomial.

1.

2.
Closing (2 minutes)

Factoring is the reverse process of multiplication.
Look for a GCF first when you are factoring a polynomial.
Keep factoring until all the factors are prime.
Factor the difference of squares as
Factor as .
Factor as .

Exit Ticket (4 minutes)
Lesson 1: Multiplying and Factoring Polynomial Expressions

Exit Ticket

When you multiply two terms by two terms you should get four terms. Why is the final result when you multiply two binomials sometimes only three terms? Give an example of how your final result can end up with only two terms.
Exit Ticket Sample Solutions

When you multiply two terms by two terms you get four terms. Why does the final result when you multiply two binomials sometimes come out with three terms? Give an example of how your final result can be two terms.

Often when you multiply two binomials, each has a term with the same variable, say, \(x\), and two of the terms combine to make one single -term. If the two terms combine to make zero there will be only two of the four terms left. For example:

Problem Set Sample Solutions

1. For each of the following factor out the greatest common factor:

1. 

2. 

3. 

4. 

5. 

3. Multiply:

1. 

2. 

3. 

4. 

4. Find the dimensions:
The measure of a side of a square is units. A new square is formed with each side units longer than the original square’s side. Write an expression to represent the area of the new square. (Hint: Draw the new square and count the squares and rectangles.)

Original square

5. In the accompanying diagram, the width of the inner rectangle is represented by and its length by . The width of the outer rectangle is represented by and its length by .

2. Find the area of the larger rectangle.

   Find the area of the larger (outer) rectangle by multiplying the binomials:

3. Find the area of the smaller rectangle.

   Find the area of the smaller (inner) rectangle by multiplying:

4. Express the area of the pink shaded region as a polynomial in terms of . (Hint: You will have to add or subtract polynomials to get your final answer.)

   Subtract the area of the smaller from the area of the larger rectangle:

   , which is the area of the pink shaded region.
Lesson 2: Multiplying and Factoring Polynomial Expressions

Classwork

Example 1: Using a Table as an Aid

Use a table to assist in multiplying

Exercise 1

Use a table to aid in finding the product of

Discussion

POLYNOMIAL EXPRESSION: A polynomial expression is either:

- a numerical expression or a variable symbol, or
- the result of placing two previously generated polynomial expressions into the blanks of the addition operator (___ + ___) or the multiplication operator (___ × ___).

Exercises 2–6
Multiply the following binomials and put the answers in standard form: , where , , and are constants. Use your answers to populate the table. (The first row is started for you.)

1.

Describe any patterns you noticed as you worked.
Exercises 7–10

Factor the following quadratic expressions.
Example 3: Quadratic Expressions

If the leading coefficient for a quadratic expression is not 1, the first step in factoring should be to see if all the terms in the expanded form have a common factor. Then after factoring out the greatest common factor, it may be possible to factor again. A polynomial is factored completely over the integers when all of its factors are prime.

For example: To factor \((ax^2 + bx + c)\) completely:

The GCF of the expression is \(a\), so factor out the GCF:

Now factor the difference of squares:

Another example: Follow the steps to factor \((ax^2 + bx + c)\) completely.

a. First factor out the GCF: *(Remember: When you factor out a negative number all the signs on the resulting factor will change.)*

1. Now look for ways to factor further: *(Notice the quadratic expression will factor.)*
Problem Set

8. Factor these trinomial as the product of two binomials and check your answer by multiplying:

2.

3.

4.

Factor completely:

5.

6.

7.

The parking lot at Gene Simon’s Donut Palace is going to be enlarged, so that there will be an additional 30 ft. of parking space in the front and 30 ft. on the side of the lot. Write an expression in terms of \( x \) that can be used to represent the area of the new parking lot.

Explain how your solution is demonstrated in the area model.
Lesson 3: Advanced Factoring Strategies for Quadratic Expressions

Classwork

Opening Exercise

Carlos wants to build a sandbox for his little brother. He is deciding between a square sandbox with side length that can be represented by \( x \) units or a rectangular sandbox with a length \( x + 1 \) unit more than the side of the square and width \( x - 1 \) unit less than the side of the square.

Carlos thinks the areas should be exactly the same since one unit is just moved from one side to the other.

5. Do you agree that the two areas should be the same? Why or why not?

6. How would you write the expressions that represent the length and width of the rectangular sandbox in terms of the side length of the square?

7. If you use the expressions for length and width represented in terms of the side length of the square, can you then write the area of the rectangle in the same terms?
8. How can this expression be seen as the product of a sum and difference, ?

9. Can you now rewrite the area expression for the rectangle as the difference of squares: ?

10. Look carefully at your answer to the last question. What does it tell you about the areas of the two shapes?

11. Can you verify that our algebra is correct using a diagram or visual display?
Example 1
In Lesson 2, we saw that factoring is the reverse process of multiplication. We factor a polynomial by reversing the distribution process.

Consider the following example of multiplication:

\[ \text{\textcolor{red}{\rightarrow}} \text{\textcolor{red}{\rightarrow}} \]

When we compare the numbers in the factored form (in red) with the numbers in the expanded form (in blue), we see that is the product of the two red numbers and is their sum. The latter is even more obvious when we look at the expanded form before the like terms are combined (in green).

[Note: When the -term coefficient is we usually do not write it out algebraically, but it is actually there as a coefficient. Point that out here to prepare students for the next example where the -term coefficients in the factors are not both.]

- Can you explain why that relationship exists between the red numbers and the blue numbers?

Example 2
Now compare the expansion of this binomial product to the one above:

\[ \text{\textcolor{red}{\rightarrow}} \text{\textcolor{red}{\rightarrow}} \]

In the expression lying between the two arrows (before the like terms are combined), we can see the coefficients of the “split” linear terms. Also notice that for this example we have coefficients on both -terms in the factors, and that one of the coefficients is not. We have and (bold) as the factors of the leading coefficient in the expanded form and and as the factors of the constant term. Get ready for quadratic expressions in factored form where neither of the -term coefficients are.

1. How is this product different from the first example? How is it similar?

2. Why are the green numbers different in the two examples?
Now that we have four different numbers (coefficients) in each form of the expression, how can we use the numbers in the expanded form of the quadratic expression on the right to find the numbers in the factors on the left?

Now we need to place those numbers into the parentheses for the factors so that the product matches the expanded form of the quadratic expression. Here is a template for finding the factors using what we call the product-sum method:

\[(__ \quad __)(__ \quad __)\] [We have four number places to fill in this factor template.]

\[(__ \quad __)(__)\] [We know that the __ and __ are the correct factors for __, so we start there.]

\[(__)(__)\] [We know that __ and __ are the only factors of __, with the __ opposite the __ so that the distribution process gives us one product.]

\[(__)(__)\] [Finally, we know, at least for this example, that all the numbers are positive.]

Example 3
Now try factoring a quadratic expression with some negative coefficients:

\[(__ \quad __)(__ \quad __)\] [We have four number places to fill in this factor template.]

\[(__ \quad__)\] [We know that the __ and __ or __ and __ are the only possible factors for the constant term, __, so we start there. Try and to start, if that does not work we will go back and try __ and __. We know that only one of the numbers can be negative to make the product negative.]

\[(__)(__)\] [We know that __ and __ are the only factors of __. We also know that both of these are positive (or both negative). But we do not know which positions they should take so will try both ways to see which will give a sum of __.]

\[(__)\] [Finally, we determine the two signs needed to make the final product __.]

Exercises 1–6
Factor the expanded form of these quadratic expressions. Pay particular attention to the negative/positive signs.
[Notice that there is a \( a \) as a coefficient in this one.]

[This might be a challenge. If it takes too long, try the next one.]

[This one has a negative on the leading coefficient.]

[We need to try one with fractions, too.]
Problem Set

Factor the following quadratic expressions.

2. [This one is tricky, but look for a special pattern.]
Lesson 4: Advanced Factoring Strategies for Quadratic Expressions

Classwork

Opening Exercises

Factor the following quadratic expressions:

Example 1: Splitting the Linear Term

How might we find the factors of $x^2 + bx + c$?

(a) Consider the product:

[Discuss the possibility that $a$ and $c$ are also multiplied when factoring examples where the leading coefficient is 1.]

List all possible factor pairs of: $(1, -36)$, $(-1, 36)$, $(2, -18)$, $(-2, 18)$, $(3, -12)$, $(-3, 12)$, $(4, -9)$, $(-4, 9)$, $(-6, 6)$

Find the pair that satisfies the requirements of the product-sum method (i.e., a pair of numbers whose product equals $ac$ and whose sum is $b$).

Rewrite the expression with the same first and last term but with an expanded $b$ term using that pair of factors as coefficients.

We now have four terms that can be entered into a tabular model or factored by grouping.

Factoring by grouping: Take the four terms above and pair the first two and the last two. This makes two groups:

[Form two groups by pairing the first two and the last two.]

[Factor out the GCF from each pair.]
The common binomial factor is now visible as a common factor of each group. Now rewrite by carefully factoring out the common factor, from each group:

(Note: We can factor difficult quadratic expressions, like , using the tabular model method or by splitting the linear term algebraically. Try both ways to see which one works best for you.)

**Exercises 1–4**

Factor the following expressions using your method of choice. After factoring each expression completely, check your answers using the Distributive Property. Remember: Always to look for a GCF prior to trying any other strategies.

2.

The area of a particular triangle can be represented by . What are its base and height in terms of ?
Problem Set

5. Factor Completely:

8. 

9. 

10. 

11. 

12. [Hint: Look for a GCF first.]

The area of the rectangle below is represented by the expression \( \text{square units} \). Write two expressions to represent the dimensions, if the length is known to be twice the width.

Two mathematicians are neighbors, with each owning a separate rectangular plot of land, which share a boundary and have the same dimensions. They agree that each has an area of \( \text{square units} \). One mathematician sells his plot to the other. The other wants to put a fence around the perimeter of his new larger plot of land. How many linear units of fencing will he need?

(Note: This question has two correct approaches and two different correct solutions. Can you find them both?)
Lesson 5: The Zero-Product Property

Classwork

Opening Exercise

Consider the equation . What values of , , , and would make the equation true?

Exercises 1–4

Find values of and that satisfy each of the following equations. (There may be more than one correct answer.)
Example 1

For each of the related questions below use what you know about the Zero-Product Property to find the answers.

5. The area of a rectangle can be represented by the expression, . Write each dimension of this rectangle as a binomial, and then write the area in terms of the product of the two binomials.

6. Can we draw and label a diagram that represents the rectangle’s area?
7. Suppose the rectangle’s area is known to be \( \text{square units} \). Can you find the dimensions in terms of \( ? \)

8. Rewrite the equation so that it is equal to zero and solve.

9. What are the actual dimensions of the rectangle?

10. If a smaller rectangle, which can fit inside the first rectangle, has an area that can be expressed by the equation \( \text{}. \) What are the dimensions of the smaller rectangle?

11. What value for \( \) would make the smaller rectangle have an area of \( \text{ that of the larger}?\)
Exercises 5–8

Solve. Show your work:

Problem Set

1.
Lesson 6: Solving Basic One-Variable Quadratic Equations

Classwork

Example 1

A physics teacher put a ball at the top of a ramp and let it roll down toward the floor. The class determined that the height of the ball could be represented by the equation, , where the height is measured in feet from the ground and time in seconds.

3. What do you notice about the structure of the quadratic expression in this problem? How can this structure help us when we apply this equation?

4. In the equation, explain what the represents.

5. Explain how you would use the equation to determine the time it takes the ball to reach the floor.

6. Now consider the two solutions for . Which one is reasonable? Does the final answer make sense based on this context? Explain.

Example 2
Lord Byron is designing a set of square garden plots so some peasant families in his kingdom can grow vegetables. He has enough land to make as many gardens of whatever size he wishes. The minimum size for a plot recommended for vegetable gardening is at least 2 m on a side. Lord Byron has enough space around the castle to make bigger plots. He decides that one side will be the minimum of 2 m plus an as yet undecided length, $x$.

(b) What expression can represent the area of one individual garden based on the undecided additional length, $x$?

(c) There are $n$ families in the kingdom who are interested growing vegetables in the gardens. What equation can represent the total area, $A$, of the gardens?

(d) If the total area available for the gardens is 1000 sq m, what are the dimensions of each garden?

(e) Find both values for $x$ that make the equation true (the solution set). What value of $x$ will Lord Byron need to add to the 2 m?

Exercises 1–6

Solve each equation. Some of them may have radicals in their solutions.
Analyze the solutions for Exercises 2–4. Notice how the questions all had as a factor but each solution was different (radical, mixed number, whole number). Explain how the structure of each expression affected each problem-solution pair.
Peter is a painter and he wonders if he would have time to catch a paint bucket dropped from his ladder before it hits the ground. He drops a bucket from the top of his 9-foot ladder. The height, $h$, of the bucket during its fall can be represented by the equation, $h = \text{height}$, where the height is measured in feet from the ground, and the time since the bucket was dropped is measured in seconds. After how many seconds does the bucket hit the ground? Do you think he could catch the bucket before it hits?
Problem Set

- Factor completely:

Solve:

Mischief is a Toy Poodle who competes with her trainer in the agility course. Within the course, Mischief must leap through a hoop. Mischief’s jump can be modeled by the equation $h = -16t^2 + vt + h_0$, where $h$ is the height of the leap in feet and $t$ is the time since the leap, in seconds. At what values of $t$ does Mischief start and end the jump?
Lesson 7: Creating and Solving Quadratic Equations in One Variable

Classwork

Opening

The length of a rectangle is \( \text{in.} \) more than twice a number. The width is \( \text{in.} \) less than the same number. The perimeter of the rectangle is \( \text{in.} \). Sketch a diagram of this situation and find the unknown number.

Example 1

The length of a rectangle is \( \text{in.} \) more than twice a number. The width is \( \text{in.} \) less than the same number. If the area of the rectangle is \( \text{sq in.} \), find the unknown number.

Example 2

A picture has a height that is \( \text{its width.} \) It is to be enlarged so that the ratio of height to width remains the same but the area is \( \text{sq in.} \). What are the dimensions of the enlargement?
Exercises 1–6

Solve the following problems. Be sure to indicate if a solution is to be rejected based on the contextual situation.

The length of a rectangle cm more than times its width. If the area of the rectangle is cm$^2$, find the width.

The ratio of length to width in a rectangle is . Find the length of the rectangle when the area is sq in.

One base of a trapezoid is in. more than twice the length of the second base. The height of the trapezoid is two in. less than the second base. If the area of the trapezoid is in$^2$, find the dimensions of the trapezoid. [Note: The area of a trapezoid: .]
A garden measuring $m$ by $m$ is to have a pedestrian pathway that is $m$ meters wide installed all the way around it, increasing the total area to $sq\ m$. What is the width, $m$, of the pathway?

Karen wants to plant a garden and surround it with decorative stones. She has enough stones to enclose a rectangular garden with a perimeter of $ft$, and she wants the garden to cover $sq\ ft$. What will the length and width of her garden be?

Find two consecutive odd integers whose product is $99$. [Note: There are two different pairs of consecutive odd integers and only an algebraic solution will be accepted.]
Challenge:

You have a -ft. roll of chain link fencing and a large field. You want to fence in a rectangular playground area. What are the dimensions of the largest such playground area you can enclose? What is the area of the playground?

Problem Set

Solve the following problems:

1. The length of a rectangle is cm less than its width. If the area of the rectangle is cm², find the width.
The ratio of length to width (measured in inches) in a rectangle is . Find the length of the rectangle if the area is known to be sq in.

One base of a trapezoid is three times the length of the second base. The height of the trapezoid is in. smaller than the second base. If the area of the trapezoid is \( \text{in}^2 \), find the lengths of the bases and the height of the trapezoid.

A student is painting an accent wall in his room where the length of the room is ft. more than its width. The wall has an area of \( \text{ft}^2 \). What are the length and the width, in feet?

Find two consecutive even integers whose product is . (There are two pairs and only an algebraic solution will be accepted.)
Lesson 8: Exploring the Symmetry in Graphs of Quadratic Functions

Classwork

Opening: Graph Vocabulary

2. **Axis of Symmetry** – Given a quadratic function in standard form, , the vertical line given by the graph of the equation, , is called the axis of symmetry of the graph of the quadratic function.

3. **Vertex** – The point where the graph of a quadratic function and its axis of symmetry intersect is called the vertex.

4. **End Behavior of a graph** – Given a quadratic function in the form (or ), the quadratic function is said to open up if and open down if .
   a. If , then has a minimum at the -coordinate of the vertex, i.e., is decreasing for -values less than (or to the left of) the vertex, and is increasing for -values greater than (or to the right of) the vertex.
   a. If , then has a maximum at -coordinate of the vertex, i.e., is increasing for -values less than (or to the left of) the vertex, and is decreasing for -values greater than (or to the right of) the vertex.
Exploratory Exercise 1

Below are some examples of curves found in architecture around the world. Some of these might be represented by graphs of quadratic functions. What are the key features these curves would have in common with a graph of a quadratic function?

St. Louis Arch

Bellos Falls Arch Bridge

Arch of Constantine

Roman Aqueduct

The photographs of architectural features above MIGHT be closely represented by graphs of quadratic functions. Answer the following questions based on the pictures.

3. How would you describe the overall shape of a graph of a quadratic function?

4. What is similar or different about the overall shape of the above curves?

IMPORTANT: Many of the photographs in this activity cannot actually be modeled with a quadratic function but rather are cantenary curves. These are very “quadratic-like” and can be used for our exploration purposes as they display many of the same features, including the symmetry we are exploring in this lesson.

Exploratory Exercise 2
Use the graphs of quadratic functions and to fill in the table and answer the questions on the following page.
Use your graphs and tables of values from the previous page to fill in the blanks or answer the questions for each below:

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<td>2</td>
<td>Vertex</td>
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<tr>
<td>3</td>
<td>Sign of the leading coefficient</td>
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<td></td>
<td><strong>Vertex represents a minimum or maximum?</strong></td>
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<td>--------------------------------------------</td>
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<tr>
<td>5</td>
<td><strong>Points of Symmetry</strong></td>
<td>Find and</td>
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<td>Is greater than or less than? Explain</td>
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<tr>
<td>6</td>
<td><strong>Increasing and Decreasing Intervals</strong></td>
<td>On what intervals of the domain is the function depicted by the graph increasing?</td>
</tr>
<tr>
<td>7</td>
<td><strong>Average Rate of Change on an Interval</strong></td>
<td>What is the average rate of change for the following intervals:</td>
</tr>
<tr>
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</table>
Understanding the symmetry of quadratic functions and their graphs (Look at row 5 in the chart and the tables).

12. What patterns do you see in the tables of values you made next to Graph and Graph?

Finding the vertex and axis of symmetry (Look at rows 1 and 2 of the chart.)

13. How can we know the -coordinate of the vertex by looking at the -coordinates of the zeros (or any pair of symmetric points)?

Understanding end behavior (Look at rows 3 and 4 of the chart.)

14. What happens to the -values of the functions as the -values increase to very large numbers? What about as the -values decrease to very small numbers (in the negative direction)?

15. How can we know whether a graph of a quadratic function will open up or down?

Identifying intervals on which the function is increasing (or decreasing) (Look at row 6 in the chart).
16. Is possible to determine the exact intervals that a quadratic function is increasing or decreasing just by looking at a graph of the function?

Computing average rate of change on an interval (Look at row 7 in the chart).

17. Explain why the average rate of change over the interval for Graph was zero.

18. How are finding the slope of a line and finding the average rate of change on an interval of a quadratic function similar? How are they different?

Finding a unique quadratic function

19. Can you graph a quadratic function if you don’t know the vertex? Can you graph a quadratic function if you only know the -intercepts?
20. Remember that we need to know at least two points to define a unique line. Can you identify a unique quadratic function with just two points? Explain.

21. What is the minimum number of points you would need to identify a unique quadratic function? Explain why?

Exploratory Exercise 3
Below you see only one side of the graph of a quadratic function. Complete the graph by plotting three additional points of the quadratic function. Explain how you found these points then fill in the table on the right.

2. What are the coordinates of the -intercepts?
3. What are the coordinates of the -intercept?

4. What are the coordinates of the vertex? Is it a minimum or a maximum?

5. If we knew the equation for this curve, what would the sign of the leading coefficient be?

6. Verify that the average rate of change for interval, , is . Show your steps.

7. Based on your answer to #6 in the table for Exploratory Challenge 2, what interval would have an average rate of change of ? Explain.
Problem Set

7. Khaya stated that every \( y \)-value of the graph of a quadratic function has two different \( x \)-values. Do you agree or disagree with Khaya? Explain your answer.

Is it possible for the graphs of two DIFFERENT quadratic functions to each have \( x=1 \) as its line of symmetry and both have a maximum at \( x=1 \)? Explain and support your answer with a sketch of the graphs.

Consider the key features discussed in this lesson for the four graphs of quadratic functions below. (Key features include: \(-\)intercepts, \(-\)intercept, line of symmetry, vertex, leading coefficient, increasing/decreasing intervals, and end-behavior)

Graph  | Graph  | Graph  | Graph

22. Which key features of a quadratic function do graphs \( G_1 \) and \( G_2 \) have in common? Which features are not shared?
23. Compare the graphs and and explain the differences and similarities between their key features.
24. Compare the graphs and and explain the differences and similarities between their key features.
25. What do all four of the graphs have in common?

Use the symmetric properties of quadratic functions to sketch the graph of the function below, given these points:
Lesson 9: Graphing Quadratic Functions from Factored Form,

Classwork

Opening Exercise
Solve the following equations:

\[(f)\]

\[(g)\]

Example 1
Consider the equation \(\) .

- Given this quadratic equation, can you find the point(s) where the graph crosses the \(-\)axis?
- How can we write a corresponding quadratic equation if we are given a pair of roots?
- In the last lesson, we learned about the symmetrical nature of the graph of a quadratic function. How can we use that information to find the vertex for the graph?
- How could we find the \(-\)intercept (where the graph crosses the \(-\)axis and where \(\)?)?
What else can we say about the graph based on our knowledge of the symmetrical nature of the graph of a quadratic function? Can we determine the coordinates of any other points?

Plot the points you know for this equation on graph paper and connect them to show the graph of the equation.

Exercise 1
Graph the following functions and identify key features of the graph.

1. b.
Example 2

A science class designed a ball launcher and tested it by shooting a tennis ball straight up from the top of a 15-story building. They determined that the motion of the ball could be described by the function:

where represents the time the ball is in the air in seconds and , the height, in feet, of the ball above the ground. What is the maximum height of the ball? At what time will the ball hit the ground?

8. With a graph, we can see the number of seconds it takes for the ball to reach its peak, and also how long it takes to hit the ground. How can factoring the expression help us graph this function?

9. Once we have the function in its factored form what do we need to know in order to graph it?

10. Using the graph, at what time does the ball hit the ground?

11. Over what domain is the ball rising? Over what domain is the ball falling?
12. Using the graph, what is the maximum height the ball reaches?
Exercises 2–3

2. Graph the following functions and identify key features of the graph:
   1. b. c.

3. The science class in Example 3 made some adjustments to their ball launcher so that it could accommodate a heavier ball. They moved the launcher to the roof of a 23-story building and launched an 8.8-pound shot put straight up into the air. (Note: Olympic and high school women pound shot put in track and field competitions.) The motion is described by the following function: , with representing the height, in feet, of the shot put above the ground with respect to time in seconds. (Important: No one was harmed during this experiment!)

   1. Graph the function and identify the key features of the graph.
2. After how many seconds does the shot put hit the ground?

3. What is the maximum height of the shot put?

4. What is the value of \( \) and what does it mean for this problem?
2. Graph the following on your own graph paper and identify the key features of the graph.

5.

6.

7.

8.

9.

4. A rocket is launched from a cliff. The relationship between the height, \( h \), in feet, of the rocket and the time, \( t \), in seconds, since its launch can be represented by the following function:

\[
h(t) = -16t^2 + 128t + 64\]

where \( t \) is the time in seconds the rocket is in the air and \( h \) is the height in feet of the rocket.

1. Sketch the graph of the motion of the rocket.
2. When will the projectile hit the ground? (Note: A projectile is any object that is “projected” into the air by launching, throwing, or any other means.)
3. When will the rocket hit the maximum height?
4. What is the maximum height the rocket reaches?
5. At what height was the rocket launched?
Lesson 10: Interpreting Quadratic Functions from Graphs and Tables

Classwork

Opening Exercise: Dolphins Jumping In and Out of the Water

After watching a video clip of dolphins jumping in and out of the ocean, what do you think the graph of time vs. the height of the dolphin above and below sea level may look like?

Example 1: The Dolphin Study

In a study of the activities of dolphins, a marine biologist made a 24-second video of a dolphin swimming and jumping in the ocean with a specially equipped camera that recorded one dolphin’s position with respect to time. This graph represents a piecewise function, that is defined by quadratic functions on each interval. It relates the dolphin’s vertical distance (in feet) from the surface of the water to the time (in seconds) from the start of the video. Use the graph to answer the questions below.
13. Describe what you know for sure about the actions of the dolphin in the time interval from 0–6 sec. Can you determine the horizontal distance the dolphin traveled in that time interval? Explain why or why not.

14. Where will you find the values for which and explain what they mean in the context of this problem.

15. How long was the dolphin swimming under water in the recorded time period? Explain your answer or show your work.

16. Estimate the maximum height, in feet, the dolphin jumped in the recorded -second time period? Explain how you determined your answer.
17. Locate the point on the graph where and explain what information the coordinates of that point give you in the context of this problem.

**Example 2**

The table below represents the value of Andrew’s stock portfolio, with representing the value of the portfolio, in hundreds of dollars, and is the time, in months, since he started investing. Answer the following questions based on the table of values:

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<th>(months)</th>
<th>(hundreds of dollars)</th>
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9. What kind of function do you think this table represents? How do you know?

10. Assuming this data is in fact quadratic, how much did Andrew invest in his stock initially? Explain how you arrived at this answer.
11. What is the maximum value of his stock and how long did it take to reach the maximum value?

12. If the pattern continues to follow the quadratic trend shown above, do you advise Andrew to sell or keep his stock portfolio? Explain why.

13. How fast is Andrew’s stock value decreasing between $t$? Find another two-month interval where the average rate of change is faster than $\frac{3}{8}$ and explain why.

14. Are there other two-month intervals where the rate of change is same as $\frac{3}{8}$? Explain your answer.
Pettitte and Ryu each threw a baseball into the air.

The vertical height of Pettitte’s baseball is represented by the graph below. \( h(t) \) represents the vertical distance of the baseball from the ground in feet and represents time in seconds.

The vertical height of Ryu’s baseball is represented by the table values below.
represents the vertical distance of the baseball from the ground in feet and \( t \) represents time in seconds.

Use the above functions to answer the following questions.

13. Whose baseball reached the highest? Explain your answer.

14. Whose ball reached the ground fastest? Explain your answer.

15. Pettitte claims that his ball reached its maximum faster than Ryu’s? Is his claim correct or incorrect? Explain your answer.

16. Find \( v_0 \) and \( h \) values and explain what it means in the problem. What conclusion can you make based on these values? Did they throw the ball from the same place? Explain your answer.

17. Ryu claims that he can throw the ball higher than Pettitte. Is his claim correct or incorrect? Explain your answer.
Lesson 11: Completing the Square

Classwork

Opening Exercise

Rewrite the following perfect square quadratic expressions in standard form. Look for patterns in the coefficients and write two sentences describing what you notice.

<table>
<thead>
<tr>
<th>FACTORED FORM</th>
<th>WRITE THE FACTORS</th>
<th>DISTRIBUTED</th>
<th>STANDARD FORM</th>
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</thead>
<tbody>
<tr>
<td>Example:</td>
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</table>

Example 1

Now try working backwards. Rewrite the following standard form quadratic expressions as perfect squares.

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<tr>
<th>STANDARD FORM</th>
<th>FACTORED FORM</th>
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<tbody>
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</tbody>
</table>
Example 2
Find an expression equivalent to that includes a perfect square binomial.

Exercises 1–10
Rewrite each expression by completing the square:
Problem Set

2.
Which of these expressions would be most easily rewritten by factoring? Justify your answer.
Lesson 12: Completing the Square

Classwork

Opening Exercise

Rewrite each expression by completing the square.

8.

9.

Example 1

Now complete the square for:

Example 2
**Business Application Vocabulary**

**Unit Price (Price per Unit):** The price per item a business sets to sell its product, sometimes represented as a linear expression.

**Quantity:** The number of items sold, sometimes represented as a linear expression.

**Revenue:** The total income based on sales (but without considering the cost of doing business).

**Unit Cost (Cost per Unit) or Production Cost:** The cost of producing one item, sometimes represented as a linear expression.

**Profit:** The amount of money a business makes on the sale of its product. Profit is determined by taking the total revenue (the quantity sold multiplied by the price per unit) and subtracting the total cost to produce the items (the quantity sold multiplied by the production cost per unit): \( \text{Profit} = \text{Total Revenue} - \text{Total Production Cost} \).

The following business formulas will be used in this and the remaining lessons in the module:

- Total Production Costs = \( (\text{cost per unit})(\text{quantity of items sold}) \)
- Total Revenue = \( (\text{price per unit})(\text{quantity of items sold}) \)
- \( \text{Profit} = \text{Total Revenue} - \text{Total Production Costs} \)

Now solve the following problem:

A certain business is marketing their product and has collected data on sales and prices for the past few years. They determined that when they raised the selling price of the product, the number of sales went down. The cost of producing a single item is $10.

12. Using the data they collected in this table, determine a linear expression to represent the quantity sold, \( q \).

13. Now find an expression to represent the profit function, \( \text{Profit} \).
Exercises 1–6

For Exercises 1-5, rewrite each expression by completing the square.

5.

6.

7.

8.
10. A fast food restaurant has determined that their price function is $p(x) = \frac{1}{4}x^2 + 2x + 10$, where $x$ represents the number of hamburgers sold.

5. The cost of producing $x$ hamburgers is determined by the expression $c(x) = x^2 + 5x + 10$. Write an expression representing the profit for selling $x$ hamburgers.

6. Complete the square for your expression in part (a) to determine the number of hamburgers that need to be sold to maximize the profit, given this price function.

Problem Set
Rewrite each expression by completing the square.

18.

19.

20.
21.

22.
Lesson 13: Solving Quadratic Equations by Completing the Square

Classwork

Opening Exercise

Solve the equation for:

Rewrite the expression by completing the square:

Example 1

Solve for:
Example 2
Now try this one:

Exercises 1–4
Solve each equation by completing the square.

(h)
Problem Set

Solve each equation by completing the square.

\[
p^2 + 3p = 8
\]
Lesson 14: Deriving the Quadratic Formula

Classwork

Opening Exercises

Solve for \( x \) by completing the square:

Solve for \( y \) by completing the square:

Discussion

Solve

Exercises 1–4
Use the quadratic formula to solve each equation.

(i) (watch the negatives)

(Did you remember the negative?)

Exercise 5

Solve these quadratic equations, using a different method for each: solve by factoring, solve by completing the square, and solve using the quadratic formula. Before starting, indicate which method you will use for each.
Problem Set

Use the quadratic formula to solve each equation.

- Solve for .

Solve for .

Solve for .
Lesson 15: Using the Quadratic Formula

Classwork

Opening Exercise

Solve:

Exercises 1–5

Solve the following equations using the quadratic formula:

2. 

3.
Exercises 6–10

For Exercises 6–9, without solving, determine the number of real solutions for each quadratic equation:

7.

8.

9.

10.
State whether the discriminant of each quadratic equation is positive, negative, or equal to zero on the line below the graph. Then identify which graph matches the discriminants below:

<table>
<thead>
<tr>
<th>Discriminant A:</th>
<th>Discriminant B:</th>
<th>Discriminant C:</th>
<th>Discriminant D:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Graph #: ______</td>
<td>Graph #: ______</td>
<td>Graph #: ______</td>
<td>Graph #: ______</td>
</tr>
</tbody>
</table>

**Problem Set**

Without solving, determine the number of real solutions for each quadratic equation.

b.

c.
d.

e.
Lesson 16: Graphing Quadratic Functions from the Vertex Form,

Classwork

Opening Exercise
Graph the equations:
, , and on the interval

Exercises 1–2

Without graphing, state the vertex for each of the following quadratic equations:

18.

19.

20.

Write a quadratic equation whose graph will have the given vertex:
Problem Set

14. Find the vertex of the graphs of the following quadratic equations.

24.

25.

Write quadratic equations representing the function with the following vertex. Use a leading coefficient other than 1.
26.

27.

Using vocabulary from this lesson, describe the relationship between the quadratic equations and .
Lesson 17: Graphing Quadratic Functions from the Standard Form,

Classwork

Opening Exercise

A high school baseball player throws a ball straight up into the air for his math class. The math class was able to determine that the relationship between the height of the ball and the time since it was thrown could be modeled by the function, \( f(x) = -16x^2 + 96x + 100 \), where \( x \) represents the time (in seconds) since the ball was thrown and \( f(x) \), the height (in feet) of the ball above the ground.

6. What does the domain of the function represent in this context?

7. What does the range of this function represent?

8. At what height does the ball get thrown?

9. After how many seconds does the ball hit the ground?

10. What is the maximum height that the ball reaches while in the air? How long will the ball take to reach its maximum height?

11. What feature(s) of this quadratic function are “visible” since it is presented in the standard form, \( f(x) = ax^2 + bx + c \)?
12. What feature(s) of this quadratic function are “visible” when it is rewritten in vertex form?

A general strategy for graphing a quadratic function from the standard form:

Look for hints in the function’s equation for general shape, direction, and -intercept.

Solve find the -intercepts by factoring, completing the square, or using the quadratic formula.

Find the vertex by completing the square or using symmetry. [Find the axis of symmetry and the -coordinate of the vertex using and the -coordinate of the vertex using .]

Plot the points you know (at least three are required for a unique quadratic function), sketch the graph of the curve that connects them, and identify the key features of the graph.

Example 1

A high school baseball player throws a ball straight up into the air for his math class. The math class was able to determine that the relationship between the height of the ball and the time since it was thrown could be modeled by the function, , where represents the time (in seconds) since the ball was thrown and , the height (in feet) of the ball above the ground.

28. What do you notice about the equation, just as it is, that will help us in creating our graph?

29. Can we factor to find the zeros of the function? If not, solve by completing the square.

30. Which will you use to find the vertex? Symmetry? Or the completed-square form of the equation?

31. Now we plot the graph of and identify the key features in the graph.
Exercises 1–5

6. Graph the equation and identify the key features.
Paige wants to start a summer lawn-mowing business. She comes up with the following profit function that relates the total profit to the rate she charges for a lawn-mowing job:

Both profit and her rate are measured in dollars. Graph the function in order to answer the following questions.

32. Graph

33. According to the function, what is her initial cost (e.g., maintaining the mower, buying gas, advertising)? Explain your answer in the context of this problem.

34. Between what two prices does she have to charge to make a profit?

35. If she wants to make $275 profit this summer is this the right business choice?
A student throws a bag of chips to her friend. Unfortunately, her friend does not catch the chips and the bag hits the ground. The distance from ground (height) for the bag of chips is modeled by the function where \( h \) is the height (distance from the ground) of the chips and \( t \) is the number of seconds the chips are in the air.

36. Graph.

37. From what height are the chips being thrown? Tell how you know.

38. What is the maximum height the bag of chips reaches while airborne? Tell how you know.

39. How many seconds after the bag was thrown did it hit the ground?

40. What is the average rate of change of height for the interval from 0 to \( \frac{1}{2} \) second? What does that number represent in terms of the context?

41. Based on your answer to part (e), what is the average rate of change for the interval from 1.5 to 2 sec?
Notice how the profit and height functions both have negative leading coefficients? Explain why this is.
Problem Set

2. Graph and identify its key features.

Graph the following equation and identify its key features.

Did you recognize the numbers in the first two problems? The equation in the second problem is the product of and the first equation. What effect did multiplying the equation by have on the graph?

Giselle wants to run a tutoring program over the summer. She comes up with the following profit function:

Between what two prices should she charge to make a profit? How much should she charge her students if she wants to make the most profit?

Doug wants to start a physical therapy practice. His financial advisor comes up with the following profit function for his business: . How much will it cost for him to start the business? What should he charge his clients to make the most profit?
Lesson 18: Graphing Cubic, Square Root, and Cube Root Functions

Classwork

Opening Exercises

Evaluate when .

Evaluate when .

Evaluate when .

Evaluate when .

Exercise 1

Use your graphing calculator to create a data table for the functions and for a variety of -values. Use both negative and positive numbers and round decimal answers to the nearest hundredth.
Exercise 2

Create the graphs of $f$ and $g$ on the same set of axes.
Exercise 3
Create data tables for $f(x)$ and $g(x)$ and graph both functions on the same set of axes. Round decimal answers to the nearest hundredth.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
<th>$g(x)$</th>
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</table>
Lesson 19: Translating Functions

Classwork

Opening Exercise

Graph each set of three functions in the same coordinate plane (on your graphing calculator or a piece of graph paper). Then explain what similarities and differences you see among the graphs:

1.  
2.  
3.  

6
Example 1

For each graph answer the following:
What is the parent function?
How does the translated graph relate to the graph of the parent function?
Write the formula for the function depicted by the translated graph.

a.

b.

c.
Exercises 1–3

For each of the following graphs, use the formula for the parent function to write the formula of the translated functions.

a.

Below is a graph of a piecewise function whose domain is . Sketch the graph of the given functions on the same coordinate plane. Label your graphs correctly.

Match the correct equation and description of the function with the given graphs.

<table>
<thead>
<tr>
<th>Graphs</th>
<th>Equation</th>
<th>Description</th>
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<tbody>
<tr>
<td>E1.</td>
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<td>D1. The graph of the parent function is translated down 3 units and left 2 units.</td>
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<tr>
<td>E2.</td>
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<tr>
<td>Equation</td>
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**E3.**

D2. The graph of the function does not have $y$-intercept.

D3. The coordinate of the $y$-intercept is $(0, 1)$ and both $y$-intercepts are positive.

D4. The graph of the function has only one $y$-intercept.
Problem Set

- Graph the functions in the same coordinate plane. Do not use a graphing calculator.

- Write a function that translates the graph of the parent function down 7.5 units and right 2.5 units.

- How would the graph of \( f(x) \) be affected if the function were transformed to \( f(x) + k \)?

- Below is a graph piecewise function whose domain is the interval \( [a, b] \). Sketch the graph of the given functions below. Label your graphs correctly.

  ![Graph Piecewise Function](image)

  [Be careful here! This one might be a challenge.]

- Study the graphs below. Identify the parent function and the transformations of that function depicted by the second graph. Then write the formula for the transformed function.
Lesson 20: Stretching and Shrinking Functions
Classwork

Opening Exercise

The graph of a quadratic function defined by $f(x)$ has been translated 5 units to the left and 3 units up. What is the formula for the function, $f(x)$, depicted by the translated graph?

Sketch the graph of the equation $f(x) = x^2 - 6x + 9$.

Example 1

Exercise 1

Complete the following to review Module 3 concepts:
12. Consider the function . Complete the table of values for . Then, graph the equation on the coordinate plane provided for part (b).

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13. Complete the following table of values for each transformation of the function . Then graph the equations , , , and on the same coordinate as the graph of . Label each graph.

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</table>
14. Describe how the graph of relates to the graph of , for each of the following cases:

1.

2.
15. Describe the transformation of the graph of \( f(x) \) that results in the graphs of \( g(x) \), \( h(x) \), and \( k(x) \), given the following formulas for each function. Then graph each function and label each graph.
16. Consider the function Complete the table of values, then graph the equation.

17. Complete the following table of values, rounding each value to the nearest hundredth. Graph the equations, and on the same coordinate plane as your graph of above. Label each graph.
18. Describe the transformations of the graph of that result in the graphs of, , and .

19. Describe how the graph of relates to the graph of , for each of the following cases:
   1.
   2.
   3.
   4.
   5. 1
Exercise 2

For each of the sets below answer the following questions:

- What are the parent functions?
- How does the translated graph relate to the graph of the parent function?
- Write the formula for the function depicted by the translated graph.
Exercise 3
Graph each set of functions in the same coordinate plane. Do not use a calculator.

15.

Problem Set

11. Graph the functions in the same coordinate plane. Do not use a calculator.
12. Explain how the graphs of functions and $f(x)$ are related.

13. Explain how the graphs of functions $g(x)$ and $f(x)$ are related.

14. Write a function, in terms of another function, such that the graph of $f(x)$ is a vertical shrink of the graph of $g(x)$ by a factor of 0.75.

15. A teacher wants the students to write a function based on the parent function $f(x)$. The graph of $g(x)$ is stretched vertically by a factor of 4 and shrunk horizontally by a factor 3. Mike wrote $g(x)$ as the new function while Lucy wrote $f(x)$. Which one is correct? Justify your answer.
16. Study the graphs of two different functions below. Which is a parent function? What is the constant value(s) multiplied to the parent function to arrive at the transformed graph? Now write the function defined by the transformed graph.

Lesson 21: Transformations of the Quadratic Parent Function,

Example 1: Quadratic Expression Representing a Function

3. A quadratic function is defined by . Write this in the completed-square (vertex) form and show all the steps.

4. Where is the vertex of the graph of this function located?

5. Look at the completed-square form of the function. Can you name the parent function? How do you know?
6. What transformations have been applied to the parent function to arrive at ? Be specific.

7. How does the completed square form relate to the quadratic parent function, ?

Example 2
The graph of a quadratic function has been translated 3 units to the right, vertically stretched by a factor of , and then moved units up. Write the formula for the function that defines the transformed graph.

Exercise 1
Sketch the graph of the following quadratic functions on the same coordinate plane, using transformations of the graph of the parent function .
No calculators this time.

1.

2.

3.
Exercises 2–4

Write a formula for the function that defines the described transformation of the graph of the quadratic parent function, .

6. 3 units shift to the right
7. Vertical shrink by a factor of 0.5
8. Reflection across the -axis
9. 4 units shift up

Then graph both the parent and the transformed functions on the same coordinate plane.
Describe the transformation of the quadratic parent function, that results in the quadratic function.
Sketch the graphs of the following functions based on the graph of the function. If necessary, rewrite some of the functions in the vertex (completed-square) form. Label your graphs.

10.

11.

12.

13.
Problem Set

- Write the function in completed square form. Describe the transformations of the graph of the parent function, , that result in the graph of .

Write the formula for the function whose graph is the graph of translated units to the right, vertically stretched by a factor of, and then translated units up.

Without using a graphing calculator, sketch the graphs of the functions below based on transformations of the graph of the parent function . Use your own graph paper and label your graphs.

42.

43.

44.

45.

46.
Lesson 22: Comparing Quadratic, Square Root, and Cube Root Functions Represented in Different Ways

Classwork

Opening Exercise

Populate the table on the right with values from the graph.
Exercises 1–3

Solve each problem and show or explain how you found your answers:

Xavier and Sherleese each threw a baseball straight up into the air. The relationship between the height (distance from the ground in feet) of Sherleese’s ball with respect to the time since it was thrown, in seconds, is given by the function:

The graph of the height as a function of time of Xavier’s ball is represented below:

Xavier claims that his ball went higher than Sherleese’s. Sherleese disagrees. Answer the questions below, then defend your answers mathematically by comparing the features found in the equation to those in the graph.

47. Who is right?

48. For how long is each baseball airborne?
49. Construct a graph of Sherleese’s throw as a function of time (t) on the same set of axes as the graph of Xavier’s, and use the graph to support your answers to parts (a) and (b).
At an amusement park, there is a ride called The Centre. The ride is a cylindrical room that spins as the riders stand along the wall. As the ride reaches maximum speed, riders are pinned against the wall and are unable to move. The model that represents the speed necessary to hold the riders against the wall, uses the function,

\[ v = \frac{\omega r}{2} \]

where \( v \) = required speed of the ride (in meters per second) and \( r \) = the radius (in meters) of the ride.

In a competing ride, The Spinner, a car spins around a center post. The measurements in the table below show the relationship between the radius \( r \) of the spin, in meters, and the speed \( v \) of the car, in m/sec.

<table>
<thead>
<tr>
<th>(meters)</th>
<th>(meters per second)</th>
</tr>
</thead>
<tbody>
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<td></td>
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</tbody>
</table>

Due to limited space at the carnival, the maximum spin radius of rides is 4 meters. Assume that the spin radius of both rides is exactly 4 meters. If riders prefer a faster spinning experience, which ride should they choose? Show how you arrived at your answer.
The growth of a Great Dane puppy can be represented by the graph below, where \( y \) represents the shoulder height (in inches) and \( x \) represents the puppy’s age (in months).

![Graph of Great Dane growth](image)

The growth of a lion cub can be modeled by the function represented in the table below.

<table>
<thead>
<tr>
<th>(months since birth)</th>
<th>(height in inches)</th>
</tr>
</thead>
</table>


50. Which animal has the greater shoulder height at birth?

51. Which animal will have the greater shoulder height at 3 years of age (the age each animal is considered full grown)?

52. If you were told that the domain for these functions is the set of all real numbers, would you agree? Why or why not?
Lesson 23: Modeling with Quadratic Functions

Opening Exercise: The Mathematics of Objects in Motion

Read the following explanation of the Mathematics of Objects in Motion:

Any object that is free falling, or projected into the air without a power source, is under the influence of gravity. Because of the constant force of earth’s gravity (represented by \( g \)), all free-falling objects on Earth accelerate toward the center of the earth (downward) at a constant rate (\( \frac{ft}{sec^2} \) or \( \frac{m}{sec^2} \)). That acceleration rate is included in the physics formula used for all objects in a free falling motion. It represents the relationship between the height of the object (distance from Earth) with respect to the time that has passed since the launch or fall began. That formula is:

\[
\text{height} = -\frac{1}{2}gt^2 + \text{initial height}
\]

For this reason, the leading coefficient for a quadratic function that models the position of a falling, launched, or projected object must either be 0 or 1. Physicists use mathematics to make predictions about the final outcome of a falling or projected object.

The mathematical formulas (equations) used in physics commonly use certain variables to indicate quantities that are most often used for motion problems. For example the following are commonly used variables for an event that includes an object that has been dropped or thrown:

- \( h \) is often used to represent the function of height (how high the object is above the Earth in feet or meters)
- \( t \) is used to represent the time (number of seconds) that have passed in the event
- \( v \) is used to represent velocity (the rate at which an object changes position in feet/sec or meters/sec)
- \( d \) is used to represent the object’s change in position, or displacement (how far the object has moved in feet or meters)

We often use subscripts with the variables, partly so that we can use the same variables multiple times in a problem without getting confused, but also to indicate the passage of time. For example:

- \( v_0 \) indicates the initial velocity (velocity at 0 seconds).
- \( h_0 \) tells us the height of the object at 0 seconds, or the initial position.

So putting all that together, we have a model representing the motion of falling or thrown objects, using U.S. standard units, as a quadratic function:

\[
\text{height} = -\frac{1}{2}gt^2 + \text{initial height}
\]

where \( h \) represents the height of the object in feet (distance from the Earth), and \( t \) is the number of seconds the object has been in motion, or if units are metric. [Note: The negative sign in front of the \( \frac{1}{2} \) indicates the downward pull of gravity. We are using a convention for quantities with direction here: upward is positive – downward is negative.]

where everything else is the same but now the height of the object is measured in meters and the velocity in meters per second.
These physics functions can be used to model many problems presented in the context of free falling or projected objects (objects in motion without any inhibiting or propelling power source, such as a parachute or an engine).
Example 1

Use the information in the Opening to answer the following questions:

Chris stands on the edge of a building at a height of \( h \) ft. and throws a ball upward with an initial velocity of \( v \) ft. per second. The ball eventually falls all the way to the ground. What is the maximum height reached by the ball? After how many seconds will the ball reach its maximum height? How long will it take the ball to reach the ground?

- What units will we be using to solve this problem?
- What information from the contextual description do we need to use in the function equation?
- What is the maximum point reached by the ball? After how many seconds will it reach that height? Show your reasoning. [Note: Students may try to complete the square for this function. The calculations are very messy but the results will be the same: \( \text{[value]} \).]
How long will it take the ball to land on the ground after being thrown? Show your work.

Graph the function of the height of the ball in feet to the time in seconds. Include and label key features of the graph such as the vertex, axis of symmetry, and -intercepts and -intercepts.
Example 2 (10 minutes)

Read the following information about Business Applications:

Many business contexts can be modeled with quadratic functions. This is because the expressions representing price (price per item), the cost (cost per item), and the quantity (number of items sold) are typically linear. The product of any two of those linear expressions will produce a quadratic expression that can be used as a model for the business context. The variables used in business applications are not as traditionally accepted as they are in physics but there are some obvious reasons to use for cost, for price, and for quantity (all lower-case letters). For total production cost we often use for the variable, for total revenue, and for total profit (all upper-case letters). You have seen these formulas in previous lessons but it is a good idea to review them here since we use them in the next two lessons:

Business Application Vocabulary

Unit Price (Price per Unit): The price per item a business sets to sell its product, sometimes represented as a linear expression.

Quantity: The number of items sold, sometimes represented as a linear expression.

Revenue: The total income based on sales (but without considering the cost of doing business).

Unit Cost (Cost per Unit) or Production Cost: The cost of producing one item, sometimes represented as a linear expression.

Profit: The amount of money a business makes on the sale of its product. Profit is determined by taking the total revenue (the quantity sold multiplied by the price per unit) and subtracting the total cost to produce the items (the quantity sold multiplied by the production cost per unit): Profit = Total Revenue – Total Production Cost.

The following business formulas will be used in this lesson:

Total Production Costs = (cost per unit)(quantity of items sold)

Total Revenue = (price per unit)(quantity of items sold)

Profit = Total Revenue – Total Production Costs

Now answer the following questions related to a business problem:

A theater decided to sell special event tickets to benefit a local charity at \$10 per ticket. The theater can seat up to 500 people and they expect to be able to sell all 500 seats for the event. To maximize the revenue for this event, a research company volunteered to do a survey to find out if they could increase the ticket price without losing revenue. The results showed that for each increase in ticket price, fewer tickets will be sold.

1. Let \( \text{price-per-ticket increases} \) represent the number of price-per-ticket increases. Write an expression to represent the expected price for each ticket.
2. Use the survey results to write an expression representing the possible number of tickets sold.

3. Using as the number of -ticket price increases and the expression representing price per ticket, write the function, , to represent the total revenue in terms of the number of -ticket price increases?

4. How many -ticket price increases will produce the maximum revenue? (i.e., what value for produces the maximum?)

5. What is the price of the ticket that will provide the maximum revenue?

6. What is the maximum revenue?

7. How many tickets will the theater sell to reach the maximum revenue?
8. How much more will the theater make for the charity by using the results of the survey to price the tickets than they would had they sold the tickets for their original price?

Exercise 1

Solve and graph the following problem:

Two rock climbers try an experiment while scaling a steep rock face. They each carry rocks of similar size and shape up a rock face. One climbs to a point ft. above the ground and the other to a place below her at ft. above the ground. The higher climber drops her rock and second later the lower climber drops his. [Note: The climbers are not vertically positioned. No climber is injured in this experiment.]

f. Define the variables in this situation and write the two functions that can be used to model the relationship between the heights, and, of the rocks, in feet, after seconds.
g. Assuming the rocks fall to the ground without hitting anything on the way, which of the two rocks will reach the ground last? Show your work and explain how you know your answer is correct.

h. Graph the two functions on the same coordinate plane and identify the key features that show that your answer to part (b) is correct. Explain how the graphs show that the two rocks hit the ground at different times.
i. Does the graph show how far apart the rocks were when they landed? Explain.

Exercise 2
Amazing Photography Studio takes school pictures and charges $10 for each class picture. The company sells an average of 20 class pictures in each classroom. They would like to have a special sale that will help them sell more pictures and actually increase their revenue. They hired a business analyst to determine how to do that. The analyst determined that for every reduction of $1 in the cost of the class picture, there would be an additional 10 pictures sold per classroom.

a. Write a function to represent the revenue for each classroom for the special sale.

b. What should the special sale price be?

c. How much more will the studio make than they would have without the sale?
Lesson 24: Modeling with Quadratic Functions

Opening Exercise

Draw as many quadratic graphs as possible through the following two points on the graph. Check with your neighbors for ideas. These points are () and ()

Example 1

Use the example with the blue points above: (), (), and (), to write the equation for the quadratic containing the three points.
Exercise 1

Write in standard form the quadratic function defined by the points , and .

Exercise 2

Louis dropped a watermelon from the roof of a tall building. As it was falling, Amanda and Martin were on the ground with a stopwatch. As Amanda called the seconds, Martin recorded the floor the watermelon was passing. They then measured the number of feet per floor and put the data collected into this table. Write a quadratic function to model the following table of data relating the height of the watermelon (distance in feet from the ground) to the number of seconds that had passed.

<table>
<thead>
<tr>
<th>Time (s)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Height</td>
<td>300</td>
<td>284</td>
<td>236</td>
<td>156</td>
</tr>
</tbody>
</table>

5. How do we know this data will be represented by a quadratic function?

6. Do we need to use all five data points to write the equation?

7. Are there any points that are particularly useful? Does it matter which we use?
8. How does this equation for the function match up with what you learned about physics in Lesson 23? Is there a more efficient way to find this equation?

9. Can you use your quadratic function to predict at what time, t, the watermelon will hit the ground (i.e., )?
Problem Set

Write a quadratic function to fit the following points, and state the $x$-values for both roots. Then sketch the graph to show that the equation includes the three points.

Write a quadratic function to fit the following points: